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Dynamics of elastic waves in two-dimensional phononic crystals with chaotic defect

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The authors study dynamics of wave function in two-dimensional phononic crystals with different billiard defects. It is found that the elastic wave is localized in the defect region with soft material. The spatial statistical properties of wave function are studied. More strikingly, they find that given the same area, the chaotic billiard contains more energy than the integrable billiards such as rectangular and circular billiards. The dependence of this “chaotic effect” on wave number k is also studied. © 2007 American Institute of Physics. [DOI: 10.1063/1.2779967]

Two-dimensional (2D) phononic crystal (PC) is a periodic elastic structure. It has attracted increasing attention in recent years\(^1\)–\(^3\) due to its potential industrial applications. It is well known that defects and/or impurities will cause wave localization, thus affect the transport behavior of the system.\(^4\) Therefore, by embedding different billiards or impurities, people expect to alter the properties of phononic transmission.

In this letter, we study the wave dynamics in the PC with defects of different geometries—stadium, rectangle, and circle. In particular, we study the energy in the defect region (\(E_b\)) and the whole PC system (\(E_{PC}\)) as a function of time by the finite-difference time-domain method.\(^5\) The purpose is to understand whether the underlying dynamics, instead of the geometry, affect the wave dynamics in phononic crystals. From the study of quantum chaos, it is well known that the statistics of wave function and energy level spacing distribution in a quantum billiard is not determined by its geometry; instead, it is determined by its underlying dynamics.\(^6\) For example, the integrable systems have the same energy level spacing distribution. Poisson distribution, no matter whether it is a circular one or a rectangular one, while chaotic systems such as the Bunimovich stadium and Sinai billiard have the Wigner distribution.\(^6\) The stadium billiard (see Fig. 1(a)) is a Hamiltonian system that was proved by Bunimovich\(^7\) to be strongly chaotic.

The equations governing the motion of lattice displacements \(u_i(r,t)\) in the inhomogeneous system are given by

\[
\rho(r) \frac{\partial^2}{\partial t^2} u_i(r,t) = \nabla_i T_{ij}(r,t) \quad (i = 1,2,3; J = 1,2, \ldots, 6),
\]

(1)

\(T_{ij}(r,t) = C_{ij}(r) S_j(r,t)\), and \(S_j(r,t) = \nabla_i u_i(r,t) \quad (I = 1,2, \ldots, 6; J = 1,2,3)\), where \(r = (x,y,z)\) (the z axis is taken to be parallel to the cylinder axis). \(\nabla\) is

\[
\nabla = \begin{bmatrix}
\partial/\partial x & 0 & 0 & \partial/\partial z & \partial/\partial y \\
0 & \partial/\partial y & 0 & \partial/\partial z & \partial/\partial x \\
0 & 0 & \partial/\partial z & \partial/\partial y & \partial/\partial x
\end{bmatrix},
\]

(2)

and \(\nabla^T\) is the transposed matrix of \(\nabla\). \(\rho(r)\) and \(C_{ij}(r)\) \((I = 1,2, \ldots, 6; J = 1,2, \ldots, 6)\) are the position-dependent mass density and elastic stiffness tensor of the system, respectively, and \(T_i(r,t)\) and \(S_j(r,t)\) are the Ith and Jth components of the stress vector and strain vector, respectively. Note that \(\rho\) and \(C_{ij}\) do not depend on \(z\) because of the homogeneity of the system along the cylinder axis. The summation over repeated indices is assumed in the present letter. The total energy density \(\varepsilon\), is given by \(\varepsilon = \varepsilon_k + \varepsilon_p\), where \(\varepsilon_k = \frac{1}{2} \rho(r) \mathbf{v}(r,t) \cdot \mathbf{v}(r,t)\) and \(\varepsilon_p = \frac{1}{2} S(r,t) C(r) S^T(r,t)\) represent the kinetic and potential energy densities, respectively, with \(\mathbf{v}(r,t)\) is the velocity vector, \(S(r,t)\) is the strain vector (\(S = [S_1,S_2,S_3,S_4,S_5,S_6]\)), \(S^T\) is the transposed vector of \(S\), and \(C(r)\) is the elastic stiffness matrix of \(C_{ij}(r)\).

The PC is a 2D square array with steel cylinders in silex background, in which the lattice constant is 6 cm and the radius of cylinders is 2.5 cm. In fact, other materials different from steel can be used as the background material. Three kinds of detected PC are studied: PC with two void cylinders [Fig. 1(c)], PC with steel billiards [Figs. 1(d)–1(f)], and PC with epoxy resin billiards [Figs. 1(g)–1(i)].

We first study the spatial distribution of wave function of different configurations. Typical wave functions in different configurations are shown in Fig. 2. Elastic wave in original PC without defect is periodically distributed. In PC with (soft material) defect, localization is clearly seen. This kind of localization phenomenon has been also observed for electromagnetic wave in photonic crystals with defect.\(^9\) In PC with steel (harder) defect, elastic wave is localized in the background material since steel is a hard material compared with the background.

To compare the wave function in nonintegrable system and integrable system more quantitatively, we calculate the two point correlation function inside the epoxy resin billiard at a given time. The results are shown in Fig. 3. The asterisks denote circular billiard and solid circles denote stadium bil-
FIG. 1. (Color online) Schematic diagrams of different PC configurations. The white region represents the silex background and the black region represents the steel rods. The blue region represents the epoxy resin. (a) Geometry of stadium billiard: half length of the straight section $a=2.97$ cm and radius of the semicircle $R=2.5$ cm. (b) Original PC. (c) PC with two void cylinders. (d) PC with steel stadium billiard. (e) PC with steel circular billiard. (f) PC with steel rectangular billiard. (g) PC with epoxy resin stadium billiard. (h) PC with epoxy resin circular billiard. (i) PC with epoxy resin rectangular billiard. Circular billiard: radius $R=3.96$ cm. Rectangular billiard: side length in the $x$ direction $L_x=7.68$ cm and side length in the $y$ direction $L_y=6.42$ cm.

As shown in Fig. 3, the difference is quite obvious. First, in the chaotic billiard, the correlation function decays very fast with the increase of the distance between the two points, whereas in the integrable case, it shows strong oscillation meaning that there exists coherent structure in the wave function in integrable billiard. Second, the correlation for chaotic billiard does not change very much with time, while it does for integrable system, which means that the spatial distribution of function in integrable systems changes with time more significantly. Equally striking is the shape of the fitted curve. The fitted curve of stadium billiard shows a similar one for quantum wave function in chaotic billiard, namely, when $k s > 0$, Corr($s$) is around zero with only a slight fluctuation.

In order to study the time dependent energy variation in space, we launch a Gaussian pulse on the left side of the PC. The central frequency of the Gaussian pulse is 500 kHz with the highest frequency of 1 MHz. Both the normalized total

![FIG. 3. (Color online) Two point spatial correlation function Corr($s$) vs $ks$ in (a) stadium billiard (solid circle and curve) and circular billiard (asterisk and dotted curve). (a) $t=180$ μs; and (b) $t=360$ μs.](image-url)
energies in the billiard ($E_b$) are shown in Fig. 4. Solid (red), dashed (blue), and thin (black) curves refer to stadium, rectangular, and circular billiards, respectively. The area of the embedded billiards remains the same for different shapes.

As shown in Fig. 4(a), $E_b$ of steel stadium billiard is almost the same as that of steel rectangular billiard and is only a little larger than that of steel circular billiard given the same billiard area. This is because steel is a hard material compared with silex; therefore, most of the energy is reflected by the hard billiard and localized in the soft background, which is shown in Figs. 2(c) and 2(d). It is the reflection of the steel billiard that plays the significant role in this case.

As to the epoxy resin billiard, it is quite different. There are two impressive phenomena shown in Fig. 4(b). One is the similarity in energy variation displayed by rectangular and circular billiards. This indicates that it is the underlying physical property of the system that determines the dynamic behavior, not the geometry. Rectangular and circular billiards contain almost the same energy in spite of the different boundary shapes. The other is that the stadium billiard possesses more energy than any of the other two. We call this characteristic “chaotic effect” in this letter. This distinction reflects the difference of nonintegrable system and integrable system.

To further investigate this chaotic effect of stadium billiard, we study different shapes of stadium billiard governed by the parameter $\gamma = a/R$ [shown in Fig. 1(a)] by fixing billiard area. Accumulated energy $[E(\gamma) = \int_0^{\infty} \mu E_{PC}(\gamma, t) \, dt]$ over the time span from 0 to 600 $\mu$s for different $\gamma$ is calculated. At $\gamma = 0$, stadium billiard degenerates into circular billiard and $E(0)$ serves as the benchmark for comparison. Figure 4(c) plots the ratio $\eta(\gamma) = [E(\gamma) - E(0)]/[E(1) - E(0)]$ at different $\gamma$. It can be seen from Fig. 4(d) that there is a remarkable distinction in $\eta(\gamma)$ between $\gamma = 0$ and $\gamma = 0.01$, though there is only a slight difference in the billiard shape. This is the manifestation of classical chaos in wave dynamics. In fact, the chaoticity of the stadium billiard can be described by the Lyapunov exponent $\lambda$, which is $\lambda = \gamma^{0.5}$. Therefore, in the case of $\gamma = 0$, the Lyapunov exponent is zero, the system is integrable, while for any nonzero $\gamma$, the Lyapunov exponent $\lambda$ is nonzero, and the underlying dynamics is completely chaotic.

Another important factor is the wave number $k$. In order to make the wave “sense” the underlying classical dynamics of the billiard, one needs to go to the so-called semiclassical limit, namely, $ka \gg 1$. This means that the wavelength must be much smaller than the typical length scale of the system, $a$ and $R$. We have calculated $E_b$ for stadium billiard and circular billiard for different $ka$, and the results are shown in Fig. 5. Solid (red) and dotted (blue) curves refer to stadium billiard and circular billiard, respectively. It is clearly seen that in the case of large wavelength such as $ka = 0.82$, the wave function does not sense the chaotic dynamics of the billiard; thus, there is very little difference between the stadium billiard and the circular billiard. Thus, $E_b$ of these two billiards are almost the same. As $ka$ is increased, $E_b$ of the stadium billiard becomes larger than that of the circular billiard and the difference in $E_b$ between these two billiards enhances correspondingly, as shown in Fig. 5(b).

In summary, we have presented a comparative study of the wave function in phononic crystals embedded with nonintegrable and integrable billiards. The energy both in the billiards shows that in the short wavelength limit, the chaotic billiard can trap more energy than the integrable counterpart does. Spatial distribution of wave function shows that elastic wave is localized in defect region with soft material. The two point correlation function demonstrates the difference of the two kinds of billiards.

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